NONSTATIONARY THERMAL STRESSES IN A HOLLOW CYLINDER WITH HEAT FLUX ACTING ON THE INSIDE SURFACE

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Thermoelastic stresses in a long hollow cylinder are considered when a uniformly distributed heat flux is applied to the inside surface and the outside surface is heat-insulated.

Calculation of the temperature and stress fields in the wall of a hollow cylinder with a short-duration heat flux applied to the inside surface is of practical interest — particularly for analysis of the thermal regime in the envelope of a tubular flashlamp [1].

An approximate solution has been given in [2] for the heat-conduction problem for a hollow cylinder; the solution is convenient when the Fourier number is small. It applies to the case in which the temperature is maintained constant at both surfaces of the cylinder. An exact solution has been given in [3] for the thermoelasticity problem for a hollow cylinder when there is convective heating of one surface and cooling of the other; an approximate solution for the initial stage of heat transfer has been obtained in [4].

Below we obtain an approximate solution of the thermoelasticity problem for a hollow cylinder heated on the inside surface by a heat flux; the solution may be used for small Fourier numbers and its region of applicability is estimated by comparison with the results of an exact solution of the problem.

We consider a long hollow cylinder $r_0 \leq r \leq R$ whose inside surface is acted on by a uniformly distributed heat flux having a power $Q(\tau)$ per unit surface area. We neglect heat transfer from the outside surface. The initial temperature of the cylinder is 0°C. The temperature distribution $t(r, \tau)$ can be found from the heat-conduction equation [5, 6]

$$\frac{1}{a}\frac{\partial t(r,\tau)}{\partial \tau} = \frac{\partial^2 t(r,\tau)}{\partial r^2} + \frac{1}{r}\frac{\partial t(r,\tau)}{\partial r} \tag{1}$$

for the initial and boundary conditions

$$t(r,0)=0,\quad -\lambda\frac{\partial t(r_0,\tau)}{\partial r} = Q(\tau),\quad \frac{\partial t(R,\tau)}{\partial r}=0. \tag{2}$$

Treating the problem as a quasi-static one, we can calculate the thermoelastic stresses in the hollow cylinder from the following formulas (see, for example, [3, 7]):

$$\sigma_\theta(r,\tau) = \frac{\alpha E}{1-\nu}\left[-t(r,\tau) + \frac{1}{r^2}\int_{r_0}^{r} t(r,\tau)\,rdr + \frac{r^2+r_0^2}{r^2(R^2-r_0^2)}\int_{r_0}^{R} t(r,\tau)\,rdr\right], \tag{3}$$

$$\sigma_r(r,\tau) = \frac{\alpha E}{1-\nu}\left[\frac{r^2-r_0^2}{r^2(R^2-r_0^2)}\int_{r_0}^{R} t(r,\tau)\,rdr - \frac{1}{r^2}\int_{r_0}^{r} t(r,\tau)\,rdr\right]. \tag{4}$$

If the ends of the cylinder are free, then

$$\sigma_z(r,\tau) = \sigma_\theta(r,\tau) + \sigma_r(r,\tau).$$

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If the ends are rigidly clamped,

$$\sigma_z(r,\ \tau) = -\alpha E t(r,\ \tau) + \nu\,[\sigma_\theta(r,\ \tau) + \sigma_r(r,\ \tau)].$$

Letting

$$\mathrm{Fo} = \frac{a\tau}{r_0^2},\quad k = \frac{R}{r_0},\quad t^*(r,\ \tau) = \frac{\lambda}{A r_0}\, t(r,\ \tau),$$

we may represent the solution of the problem (1), (2) for Q(τ) = A in the following form [8, 5]:

$$t^*(r,\ \tau) = \frac{1}{k^2-1}\left[2\mathrm{Fo} - \frac{1}{4}\left(1-2\frac{r^2}{r_0^2}\right) - k^2\left(\frac{3}{4} - \frac{\ln k}{k^2-1} + \ln\frac{r}{kr_0}\right)\right] + \pi\sum_{n=1}^{\infty}\frac{J_1(x_n)\,J_1(kx_n)}{x_n\left[J_1^2(x_n) - J_1^2(kx_n)\right]}\left[Y_1(kx_n)\,J_0\left(x_n\frac{r}{r_0}\right) - J_1(kx_n)\,Y_0\left(x_n\frac{r}{r_0}\right)\right]\exp(-x_n^2\mathrm{Fo}), \tag{5}$$

where x_n are the roots of the characteristic equation

$$J_1(x)\,Y_1(kx) - Y_1(x)\,J_1(kx) = 0. \tag{6}$$

Substituting (5) into (3), (4) and letting

$$\sigma_\theta^*(r,\ \tau) = \frac{\lambda(1-\nu)}{\alpha E A r_0}\,\sigma_\theta(r,\ \tau),\quad \sigma_r^*(r,\ \tau) = \frac{\lambda(1-\nu)}{\alpha E A r_0}\,\sigma_r(r,\ \tau),$$

we obtain the following stress distributions:

$$\sigma_\theta^*(r,\ \tau) = \frac{1}{8(k^2-1)}\left(k^2+1-3\frac{r^2}{r_0^2}+k^2\frac{r_0^2}{r^2}\right) + \frac{k^2}{2(k^2-1)}\times\left[1-\ln\left(k\frac{r_0}{r}\right) - \frac{r_0^2\left(k^2+\frac{r^2}{r_0^2}\right)}{r^2(k^2-1)}\ln k\right] + \pi\sum_{n=1}^{\infty}\frac{J_1(x_n)\,J_1(kx_n)}{x_n\left[J_1^2(x_n)-J_1^2(kx_n)\right]}\times\left\{J_1(kx_n)\,Y_0\left(x_n\frac{r}{r_0}\right) - Y_1(kx_n)\,J_0\left(x_n\frac{r}{r_0}\right) + \frac{r_0}{x_n r}\times\left[Y_1(kx_n)\,J_1\left(x_n\frac{r}{r_0}\right) - J_1(kx_n)\,Y_1\left(x_n\frac{r}{r_0}\right)\right]\right\}\exp(-x_n^2\mathrm{Fo}), \tag{7}$$

$$\sigma_r^*(r,\ \tau) = \frac{k^2-\frac{r^2}{r_0^2}}{8(k^2-1)}\left(1-\frac{r_0^2}{r^2}\right) + \frac{k^2}{2(k^2-1)}\left[\frac{r_0^2}{r^2}\cdot\frac{\left(k^2-\frac{r^2}{r_0^2}\right)}{k^2-1}\ln k - \ln\left(k\frac{r_0}{r}\right)\right] - \pi\sum_{n=1}^{\infty}\frac{J_1(x_n)\,J_1(kx_n)}{x_n^2\frac{r}{r_0}\left[J_1^2(x_n)-J_1^2(kx_n)\right]}\times\left[Y_1(kx_n)\,J_1\left(x_n\frac{r}{r_0}\right) - J_1(kx_n)\,Y_1\left(x_n\frac{r}{r_0}\right)\right]\exp(-x_n^2\mathrm{Fo}). \tag{8}$$

As we might expect, no radial stresses on the inside and outside surfaces of the hollow cylinder appear in (8). Taking into account (6) and the familiar expression

$$J_1(z)\,Y_0(z) - J_0(z)\,Y_1(z) = \frac{2}{\pi z},$$

we obtain the following expressions from (7) for the tangential stresses on the surfaces of the cylinder:

$$\sigma_\theta^*(r_0,\ \tau) = \frac{1}{4} + \frac{k^2}{2(k^2-1)}\left(1-\frac{2k^2}{k^2-1}\ln k\right) + 2\sum_{n=1}^{\infty}\frac{J_1^2(kx_n)\exp(-x_n^2\mathrm{Fo})}{x_n^2\left[J_1^2(x_n)-J_1^2(kx_n)\right]}, \tag{9}$$

$$\sigma_\theta^*(R,\ \tau) = -\frac{1}{4} + \frac{k^2}{2(k^2-1)}\left(1-\frac{2}{k^2-1}\ln k\right) + 2\sum_{n=1}^{\infty}\frac{J_1(x_n)\,J_1(kx_n)}{kx_n^2\left[J_1^2(x_n)-J_1^2(kx_n)\right]}\exp(-x_n^2\mathrm{Fo}). \tag{10}$$

When the heat flux varies with time we can obtain the stresses from the expressions for the stresses under constant heat flux by applying the Duhamel theorem (see, for example [7]).

It is difficult to use the exact solutions (5), (7)-(10) at the initial stage of development of the wall-temperature field since series convergence is slow for small Fo. Thus we make use of a solution method [5, 6] for problems of nonstationary heat conduction in which the solution is obtained in the plane of Laplace transforms by means of asymptotic expansion of the transform into a rapidly converging series; after going over to the domain of the original we find the temperature for small Fo. The solution of (1), (2) in the Laplace-transform plane is

$$\bar{t}(r,\ p)=\frac{\bar{Q}(p)}{\lambda q}\cdot\frac{K_1(qR)\,I_0(qr)+I_1(qR)\,K_0(qr)}{I_1(qR)\,K_1(qr_0)-I_1(qr_0)\,K_1(qR)},$$
$$q=\sqrt{\frac{p}{a}},\quad \bar{t}(r,\ p)=\int_0^\infty t(r,\ \tau)\exp(-p\tau)\,d\tau,\tag{11}$$
$$\bar{Q}(p)=\int_0^\infty Q(\tau)\exp(-p\tau)\,d\tau.$$

Let us apply a Laplace transform with respect to the variable τ to (3), (4). Substituting (11) into the resulting expressions, we find the following for the stresses in the transform plane:

$$\bar{\sigma}_\theta(r,\ p)=\frac{\alpha E\bar{Q}(p)}{\lambda(1-\nu)\,q}\left[-\frac{K_1(qR)\,I_0(qr)+I_1(qR)\,K_0(qr)}{I_1(qR)\,K_1(qr_0)-I_1(qr_0)\,K_1(qR)}+\right.$$
$$\left.+\frac{1}{qr}\cdot\frac{K_1(qR)\,I_1(qr)-I_1(qR)\,K_1(qr)}{I_1(qR)\,K_1(qr_0)-I_1(qr_0)\,K_1(qR)}+\frac{1}{qr}\left(\frac{r_0}{r}\right)\frac{k^2+\frac{r^2}{r_0^2}}{k^2-1}\right],\tag{12}$$

$$\bar{\sigma}_r(r,\ p)=\frac{\alpha E\bar{Q}(p)}{\lambda(1-\nu)\,q^2r}\left[-\frac{r_0}{r}\cdot\frac{k^2-\frac{r^2}{r_0^2}}{k^2-1}+\right.$$
$$\left.+\frac{I_1(qR)\,K_1(qr)-K_1(qR)\,I_1(qr)}{I_1(qR)\,K_1(qr_0)-K_1(qR)\,I_1(qr_0)}\right].\tag{13}$$

The asymptotic expansions of the modified Bessel functions for large qr, which correspond to small Fo values in the plane of originals, have the form

$$I_n(qr)=\frac{\exp(qr)}{\sqrt{2\pi qr}}\left[1-\frac{4n^2-1}{1!\,8qr}+\frac{(4n^2-1)(4n^2-3^2)}{2!\,(8qr)^2}-\ldots\right],\tag{14}$$

$$K_n(qr)=\sqrt{\frac{\pi}{2qr}}\exp(-qr)\left[1+\frac{4n^2-1}{1!\,8qr}+\frac{(4n^2-1)(4n^2-3^2)}{2!\,(8qr)^2}+\ldots\right].\tag{15}$$

In the expansions (14), (15), where $n = 0;\ 1;\ 1 \le r/r_0 \le k$, we need only allow for the first term when $|4n^2-1|/8qr \le 0.01$. In this case $2qr_0(k-1) \ge 25(k-1)r_0/r\,|4n^2-1|$ and we may assume that

$$\exp[-2qr_0(k-1)] \ll 1.\tag{16}$$

Substituting the expansions (14), (15) into (11)-(13) and taking the first term into account, we obtain the following approximate expressions:

$$\bar{t}(r,\ p)=\frac{\bar{Q}(p)}{\lambda q}\sqrt{\frac{r_0}{r}}\left[\exp\left(-qr_0\left(\frac{r}{r_0}-1\right)\right)+\exp\left(-qr_0\times\right.\right.$$
$$\left.\left.\times\left(2k-1-\frac{r}{r_0}\right)\right)\right],\tag{17}$$

$$\bar{\sigma}_\theta(r,\ p)=\frac{\alpha E\bar{Q}(p)}{\lambda(1-\nu)\,q}\left\{-\sqrt{\frac{r_0}{r}}\left[\exp\left(-qr_0\left(\frac{r}{r_0}-1\right)\right)+\right.\right.$$
$$\left.+\exp\left(-qr_0\left(2k-1-\frac{r}{r_0}\right)\right)\right]-\frac{1}{qr}\sqrt{\frac{r_0}{r}}\left[\exp\left(-qr_0\left(\frac{r}{r_0}-1\right)\right)-\right.$$
$$\left.\left.-\exp\left(-qr_0\left(2k-1-\frac{r}{r_0}\right)\right)\right]+\frac{1}{qr}\left(\frac{r_0}{r}\right)\frac{k^2+\frac{r^2}{r_0^2}}{k^2-1}\right\},\tag{18}$$

$$\bar{\sigma}_r(r,\ p)=\frac{\alpha E\bar{Q}(p)}{\lambda(1-\nu)\,q^2r}\left\{\sqrt{\frac{r_0}{r}}\left[\exp\left(-qr_0\left(\frac{r}{r_0}-1\right)\right)-\right.\right.$$

$$-\exp\left(-qr_0\left(2k-1-\frac{r}{r_0}\right)\right)\Bigg]-\frac{r_0\left(k^2-\frac{r^2}{r_0^2}\right)}{r(k^2-1)}\Bigg\}. \quad (19)$$

The values of $\overline{\sigma}(r, p)$ from (19) should vanish at the surfaces of the hollow cylinder. This condition is satisfied identically on the outside surface; it will be satisfied on the inside surface if we take (16) into account and neglect $\exp(-2qr_0(k-1))$ in (19) as compared with unity.

Allowing for (16) and going over in (17), (18) from transforms to originals, we obtain the following expressions for the temperature and the tangential stresses on the cylinder surfaces:

$$t^*(r_0, \tau)=\frac{2}{\sqrt{\pi}}\sqrt{\mathrm{Fo}}, \quad t^*(R, \tau)=4\sqrt{\frac{\mathrm{Fo}}{k}}\, i\,\mathrm{erfc}\left(\frac{k-1}{2\sqrt{\mathrm{Fo}}}\right), \quad (20)$$

$$\sigma_\theta^*(r_0, \tau)=t^*(r_0, \tau)(-1+\varepsilon), \quad \varepsilon=\frac{\sqrt{\pi\,\mathrm{Fo}}}{k^2-1}, \quad (21)$$

$$\sigma_\theta^*(R, \tau)=\frac{2\mathrm{Fo}}{k^2-1}-4\sqrt{\frac{\mathrm{Fo}}{k}}\, i\,\mathrm{erfc}\left(\frac{k-1}{2\sqrt{\mathrm{Fo}}}\right),$$

$$i^n\,\mathrm{erfc}\, z=\int_z^\infty i^{n-1}\,\mathrm{erfc}\,\xi d\xi, \quad i^0\,\mathrm{erfc}\, z=\mathrm{erfc}\, z=\frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-\xi^2)\, d\xi. \quad (22)$$

Tables of $i^n\mathrm{erfc}$ are given in [5, 6, 9]. Estimates show that we may take $\mathrm{Fo} = 2\cdot10^{-2}$ for k = 1.2-1.5 to be the upper boundary of the domain of applicability for the approximate formulas (20)-(22). On the upper boundary the temperature and stress values obtained from (20)-(22) are close to the corresponding values from the exact formulas (5), (9), (10). Thus for $\mathrm{Fo} = 2\cdot10^{-2}$, k = 1.2 the approximate and exact formulas give $t^*(r_0) = 0.159$; 0.157; $t^*(R) = 0.061$; 0.060; $\sigma_\theta^*(r_0) = -0.069$; -0.066; $\sigma_\theta^*(R) = 0.029$; 0.030. For $\mathrm{Fo} = 2\cdot10^{-2}$, k = 1.5 we obtain $t^*(r_0) = 0.159$; 0.152; $t^*(R) = 0.001$; 0.001; $\sigma_\theta^*(r_0) = 0.127$; -0.120; $\sigma_\theta^*(R) = 0.031$; 0.031. When the exact formulas were used to compute the temperature and stresses the roots of (6) were taken from [10] and the Bessel functions from the tables of [11].

Neglecting ε as compared with 1 in (21), we obtain the expression for the compressive surface stress used in [1] in estimating the critical thermal load on the tube of a flashlamp. For a pulse length $\tau \le 10^{-3}$ sec and typical dimensions of the quartz ($a = 7.5\cdot10^{-3}$ cm²/sec) tube of the lamp ($r_0 = 0.5$ cm, k = 1.3) we have $\mathrm{Fo} \le 3\cdot10^{-5}$, $\varepsilon \le 1.5\cdot10^{-2}$. Under these conditions the use of the approximation in [1] is justified. As the pulse length increases, the second term of (21), which decreases the compressive surface stress, has more and more influence. With allowance for the upper bound on Fo, when k = 1.5, 1.2 the value of $\mathrm{Fo} \le 10^{-3}$ may reach 0.20 and 0.57, respectively. Calculating the stresses from (21), (22) we see that when $\mathrm{Fo} \approx 10^{-2}$ the tensile stresses on the outside surface of the hollow cylinder are small compared with the compressive stresses on the inside surface. When $\mathrm{Fo} \approx 10^{-2}$, the tensile stresses become the same in order of magnitude as the compressive stresses and are dangerous to such materials as quartz glass whose tensile strength is many times less than the compressive strength.

Let us consider the temperature and stresses in a hollow cylinder for a variable short-term heat flux. We approximate $Q(\tau)$ by a polynomial of degree m,

$$\frac{Q(\tau)}{A}=\alpha_0+\alpha_1\tau+\ldots+\alpha_m\tau^m. \quad (23)$$

In the Laplace-transform plane the approximation (23) will have the form

$$\frac{Q(p)}{A}=\frac{0!}{p}\alpha_0+\frac{1!}{p^2}\alpha_1+\ldots+\frac{m!}{p^{m+1}}\alpha_m. \quad (24)$$

We substitute (24) into (17), (18). Going from transforms to originals, we find the following expressions for the temperature and the tangential stresses on the surfaces of the cylinder while taking (16) into account:

$$t^*(r_0, \tau)=\sqrt{\mathrm{Fo}}\,f, \quad t^*(R, \tau)=4\sqrt{\frac{\mathrm{Fo}}{k}}\,\delta, \quad (25)$$

$$\sigma_\theta^*(r_0, \tau)=-\sqrt{\mathrm{Fo}}\,f+\frac{2\mathrm{Fo}}{k^2-1}\gamma, \quad (26)$$

$$\sigma_\theta^*(R, \tau)=\frac{2\mathrm{Fo}}{k^2-1}\gamma-4\sqrt{\frac{\mathrm{Fo}}{k}}\,\delta, \quad (27)$$

$$f = \frac{\Gamma(1)}{\Gamma\left(1+\frac{1}{2}\right)}\alpha_0 + \frac{\Gamma(2)}{\Gamma\left(2+\frac{1}{2}\right)}\alpha_1\tau + \frac{\Gamma(m+1)}{\Gamma\left(m+1+\frac{1}{2}\right)}\alpha_m\tau^m =$$
$$= \frac{2}{\sqrt{\pi}}\left[\frac{0!}{1}\alpha_0 + \frac{1!\,2}{1\cdot 3}\alpha_1\tau + \ldots + \frac{m!\,2^m}{1\cdot 3\ldots(2m+1)}\alpha_m\tau^m\right],$$
$$\gamma = \alpha_0 + \frac{1}{2}\alpha_1\tau + \ldots + \frac{1}{m+1}\alpha_m\tau^m,$$
$$\delta = 0!\,\alpha_0 i\,\mathrm{erfc}\left(\frac{k-1}{2\sqrt{\mathrm{Fo}}}\right) + 1!\,4\alpha_1\tau i^3\,\mathrm{erfc}\left(\frac{k-1}{2\sqrt{\mathrm{Fo}}}\right) + \ldots +$$
$$+ m!\,4^m\alpha_m\tau^m i^{2m+1}\,\mathrm{erfc}\left(\frac{k-1}{2\sqrt{\mathrm{Fo}}}\right).$$

When $Q(\tau) = A$, $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$ according to (23). As we might expect, in this case* (25)-(27) coincide with (20)-(22), respectively.

Let us compare the temperature and stress values from (25)-(27), $Q(\tau)$ being approximated by the polynomial (23), with the results obtained when the problem is solved by the exact expression for $Q(\tau)$. As an example, let us examine an exponentially varying flux

$$\frac{Q(\tau)}{A} = \exp\left(-\frac{\tau}{\tau_0}\right). \tag{28}$$

For $\bar{Q}(p)/A = p + \tau_0^{-1}$, $r = r_0$, taking (16) into account and going over to originals in (17), (18), we obtain the following solution:

$$t^*(r_0, \tau) = \frac{2}{\sqrt{\pi}}\sqrt{\mathrm{Fo}}\sqrt{\frac{\tau_0}{\tau}}\,\varphi, \quad \varphi = \exp\left(-\frac{\tau}{\tau_0}\right)\int_0^{\sqrt{\tau/\tau_0}}\exp y^2 dy, \tag{29}$$

$$\sigma_\theta^*(r_0, \tau) = -\frac{2}{\sqrt{\pi}}\sqrt{\mathrm{Fo}}\sqrt{\frac{\tau_0}{\tau}}\,\varphi + \frac{2\mathrm{Fo}}{k^2-1}\cdot\frac{\tau_0}{\tau}\left[1-\exp\left(-\frac{\tau}{\tau_0}\right)\right]. \tag{30}$$

For $\sqrt{\tau/\tau_0} \le 2$ the values of $\int_0^{\sqrt{\tau/\tau_0}}\exp y^2 dy$ may be taken from [10]. For large $\sqrt{\tau/\tau_0}$ we have $\varphi \approx (1/2)\sqrt{\tau_0/\tau}$ [12]. We note that the temperature given by (29) for the inside surface of the cylinder coincides with the surface temperature of a plate [13] heated by an exponential flux, a condition resembling (16) being satisfied for the plate.

The heat flux $Q(\tau)$ from (28) may be approximated by a polynomial having the form (23) with the aid of a Maclaurin-series expansion of $\exp(-\tau/\tau_0)$. In this case $\alpha_m = (-1)^m(1)/(m!\tau_0^m)$ and the remainder of the series is smaller in absolute value than $[1/(m+1)!](\tau/\tau_0)^{m+1}$. Since $m = 5$ for $\tau/\tau_0 \le 1.5$, the error is less than 2%. The $t^*(r_0, \tau)/\sqrt{\mathrm{Fo}}$ values calculated from (25) for this case differ by no more than 2% from the corresponding values from (29). Comparing (26) and (30), we see that for the given case the values of γ in (26) differ by no more than 1% from the corresponding values of $\tau_0/\tau[1-\exp(-(\tau/\tau_0))]$ in (30). The temperature and stresses found with $Q(\tau)$ being approximated by a polynomial agree well with the results given by the exact expression for $Q(\tau)$.

NOTATION

λ, a, α, ν, E are the thermal conductivity coefficient, the thermal diffusivity coefficient, the coefficent of linear expansion, the Poisson ratio, and the elastic modulus;

*Since the polynomials f, γ, δ are linear in α_m, we may use the proposed method to solve the inverse problem and to find the heat flux $Q(\tau)$ in the form (23) from the prescribed time dependence of the temperature or the stresses. If at the times $\tau_1, \ldots, \tau_m$ we know the corresponding m values of the temperature on the inside surface of a hollow cylinder with prescribed λ, a, then, substituting the pairs τ_1, $t(\tau_1)$, ..., τ_m, $t(\tau_m)$ into the first formula of (25), we arrive at a system of m linear equations for determining the coefficients $\alpha_0 A, \ldots, \alpha_{m-1}A$ of (23). In like manner we can obtain the approximation coefficients in (23) by using the second formula of (25) or (26), (27), respectively, if for a hollow cylinder with prescribed geometric, thermophysical and mechanical characteristics at the times $\tau_1, \ldots, \tau_m$ we know the values of the temperature on the outside surface or the values of the stresses on one of the surfaces.

σ_θ, σ_r, σ_z	are the tangential, radial, and axial stresses;
$J_n(x)$, $Y_n(x)$	are the n-th order Bessel functions of the first and second kinds, respectively;
$I_n(x)$, $K_n(x)$	are the n-th order modified Bessel functions of the first and second kinds, respectively;
$\Gamma(x)$	is a gamma function.

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